A NOTE ON THE IRREGULAR PRIMES WITH RESPECT TO EULER POLYNOMIALS

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Abstract. An odd prime p is called irregular with respect to Euler polynomials if it divides at least one of the integers

$$2E_1(0), 2^3E_3(0), \ldots, 2^{p-2}E_{p-2}(0),$$

where $E_n(x)$ is the nth Euler polynomial.

Let $K=\mathbb{Q}(\zeta_p), K^+=\mathbb{Q}(\zeta_p)^+$ be the pth cyclotomic field and its maximal real subfield, respectively. Let S be the set of infinite places of K, T be the set of places above the prime 2, $h_{p,2}, h_{p,2}^+$ be the (S,T)-refined class number of K and K^+, respectively, as Gross in [3]. Let $h_{p,2}^-=h_{p,2}/h_{p,2}^+$ be the (S,T)-relative class number of $\mathbb{Q}(\zeta_p)$. We show that $p \mid h_{p,2}^-$ if and only if p is irregular with respect to Euler polynomials.

1. Introduction

The Bernoulli numbers $B_0, B_1, B_2, \ldots$ are given by $B_0=1$ and the recursion

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k=0 \quad \text{i.e.} \quad B_n=-\frac{1}{n+1}\sum_{k=0}^{n}\binom{n-1}{k}B_k \tag{1.1}$$
$$(n=1,2,3,\ldots).$$

The Euler numbers $E_0, E_1, E_2, \ldots$ are given by $E_0=1$ and the recursion

$$\sum_{\substack{k=0\\2\mid n-k}}^{n}\binom{n}{k}E_k=0 \quad \text{i.e.} \quad E_n=-\sum_{\substack{k=0\\2\mid n-k}}^{n}\binom{n}{k}E_k \tag{1.2}$$
$$(n=1,2,3,\ldots).$$

(See [8, Definition 1.1]).

The Bernoulli polynomials $B_k(x)$ $(k=0,1,2,\ldots)$ are defined by the generating function

$$\frac{xe^{tx}}{e^x-1}=\sum_{k=0}^{\infty}B_k(t)\frac{x^k}{k!}. \tag{1.3}$$

In particular, the numbers $B_k=B_k(0)$ $(k=0,1,2,\ldots)$ are just the Bernoulli numbers defined as in (1.1). (See [5, p. 230, Lemma 1]).

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The Euler polynomials $E_k(x)$ $(k=0,1,2,\ldots)$ are defined by the generating function

$$\frac{2e^{xt}}{e^t+1}=\sum_{k=0}^{\infty}E_k(x)\frac{t^k}{k!}. \tag{1.4}$$

Different with Bernoulli polynomials, the numbers $E_k(0)$ $(k=0,1,2,\ldots)$ are not named Euler numbers in the most literatures, while the integers $E_k=2^kE_k(1/2)$ $(k=0,1,2,\ldots)$ are called Euler numbers. For example, $E_0=1, E_2=-1, E_4=5$, and $E_6=-61$. The Euler numbers and polynomials (so called by Scherk in 1825) appear in Euler's famous book, Insitutiones Calculi Differentials (1755, pp. 487–491 and p. 522).

Kummer introduced the notion of irregular prime as follows. An odd prime p is said to be regular (with respect to Bernoulli numbers (1.1)) if p does not divide the numerator of any of the numbers $B_2, B_4, \ldots, B_{p-3}$. If p is not regular, it is called irregular. The prime 3 is irregular. (See [5, p. 233]).

The notion of irregular prime has an important application in algebraic number theory. Let $\mathbb{Q}(\zeta_p)$ be the pth cyclotomic field. Kummer proved the following result.

Theorem 1.1 (Kummer, [10, p. 62, Theorem 5.16])**.** *Let p be an odd prime and let h_p^- be the relative class number of $\mathbb{Q}(\zeta_p)$. Then $p \mid h_p^- \Leftrightarrow p$ divides the numerator of B_j for some $j=2,4,\ldots,p-3$.*

The proof of the above result is based on the following well-known decomposition of the relative class number h_p^- as the generalized Bernoulli numbers

$$h_p^-=2p\prod_{\substack{j=1\\ j\text{ odd}}}^{p-2}\left(\frac{1}{2}B_{1,\omega^j}\right).$$

(See [10, p. 43, Theorem 4.17]). Here ω is the Techmüller character (see [10, p. 51]) and for any Dirichlet character χ of conductor f, the generalized Bernoulli numbers $B_{n,\chi}$ are defined by

$$\sum_{a=1}^{f}\frac{\chi(a)te^{at}}{e^{ft}-1}=\sum_{n=0}^{\infty}B_{n,\chi}\frac{t^n}{n!}. \tag{1.5}$$

(See [10, p. 31]).

Carlitz [2] gave a similar notion of irregular prime with respect to Euler numbers (1.2), that is, a prime p is irregular with respect to Euler numbers if it divides at least one of the numbers $E_2, E_4, \ldots, E_{p-3}$. (See [2, p. 330, (3.31)]). As indicated by Carlitz, this notion has a connection with the solution of Fermat equation

$$x^p+y^p=z^p$$

by Vandiver in [9].

Analogy with Kummer and Carlitz, we here define an odd prime p is irregular with respect to Euler polynomials if it divides at least one of the

integers $2E_1(0), 2^3E_3(0), \ldots, 2^{p-2}E_{p-2}(0)$. It needs to be notice that $E_k(0) = 0$ if k is even ([8, p. 5, Corollary 1.1(ii)]) and $2^kE_k(0) \in \mathbb{Z}$ for $k \geq 1$ ([6, p. 2169, Lemma 2.1]).

We shall show the above notion of irregular also has a connection with the ideal class group of $\mathbb{Q}(\zeta_p)$. Let $K = \mathbb{Q}(\zeta_p), K^+ = \mathbb{Q}(\zeta_p)^+$ be the pth cyclotomic field and its maximal real subfield, respectively. Let S be the set of infinite places of K, T be the set of places above the prime 2, $h_{p,2}, h^+_{p,2}$ be the (S,T)-refined class number of K and K^+, respectively. For the definitions of the (S,T)-refined class numbers of global fields, we refer to Gross [3, Section 1], Aoki [1, Section 7] or [4, Section 2]. In this note, we prove the following results.

Theorem 1.2. *Let p be an odd prime and let $h^-_{p,2} = h_{p,2}/h^+_{p,2}$ be the (S,T)-relative class number of $\mathbb{Q}(\zeta_p)$. Then $p \mid h^-_{p,2} \Leftrightarrow p$ divides $2^kE_k(0)$ for some $k = 1, 3, \ldots, p-2$.*

Remark 1.3. This is an analogue of a well-known theorem for irregular primes with respect to Bernoulli numbers. (See Theorem 1.1 above).

Theorem 1.4. *There are infinitely many irregular primes with respect to Euler polynomials.*

Remark 1.5. This is also an analogue of a well-known result: "There are infinitely many irregular primes (with respect to Bernoulli numbers)." (See [10, p. 62, Theorem 5.17]).

For a primitive Dirichlet character χ with an odd conductor f, the generalized Euler numbers $E_{n,\chi}$ are defined by

$$2\sum_{a=1}^{f}\frac{(-1)^a\chi(a)e^{at}}{e^{ft}+1} = \sum_{n=0}^{\infty}E_{n,\chi}\frac{t^n}{n!}. \tag{1.6}$$

(See [4, Section 5.1]).

In what follows, the notation "$\equiv$" means $\mathrm{ord}_p(a-b) > 0$ if $a, b \in \mathbb{Q}$, that is, $a-b$ is p-integral.

To prove the main results, we need the following lemma.

Lemma 1.6. *Suppose p is an odd prime and n is an odd integer. Then*

$$E_{0,\omega^n} \equiv E_n(0) \pmod{p}.$$

Remark 1.7. This is an analogue of a well-known result for Bernoulli numbers: "Suppose n is odd, $n \not\equiv -1 \pmod{p-1}$. Then

$$B_{1,\omega^n} \equiv \frac{B_{n+1}}{n+1} \pmod{p}."$$

(See [10, p. 61, Corollary 5.15]).

2. Proofs

Proof of Proposition 1.6. By [7, Proposition 5.4], for any integers $k, n \geq 0$, we have

$$E_{n,\omega^{k-n}} = \int_{\mathbb{Z}_p} \omega^{k-n}(a)a^n d\mu_{-1}(a). \tag{2.1}$$

By [7, Proposition 2.1(1)], we have

$$E_k(0) = \int_{\mathbb{Z}_p} a^k d\mu_{-1}(a). \tag{2.2}$$

Since $\omega(a) \equiv a \pmod p$, we have

$$\omega^{k-n}(a) \equiv a^{k-n} \pmod p \text{ and } \omega^{k-n}(a)a^n \equiv a^k \pmod p. \tag{2.3}$$

From (2.1) and (2.2), we have

$$E_{n,\omega^{k-n}} - E_k(0) = \int_{\mathbb{Z}_p} (\omega^{k-n}(a)a^n - a^k) d\mu_{-1}(a).$$

By (2.3), the expression in brackets under the integral sign is $\equiv 0 \pmod p$, the theorem follows. □

Proof of Theorem 1.2. Let $K = \mathbb{Q}(\zeta_p)$, so the associate group of Dirichlet character is cyclic group of order $p-1$ generated by the Techmüller character ω, and the odd characters are ω^k, $k = 1, 3, \ldots, p-2$. By [4, Proposition 3.4],

$$\begin{aligned} h_{p,2}^- &= (-1)^{\frac{p-1}{2}} 2^{2-p} \prod_{\substack{1 \leq k < p \\ k \text{ odd}}} E_{0,\omega^k} \\ &= (-1)^{\frac{p-1}{2}} 2^{2-p} E_{0,\omega^1} E_{0,\omega^3} \cdots E_{0,\omega^{p-2}}. \end{aligned}$$

By Proposition 1.6, we have

$$h_{p,2}^- \equiv (-1)^{\frac{p-1}{2}} 2^{2-p} E_1(0) E_3(0) \cdots E_{p-2}(0) \pmod p.$$

Thus we conclude our result. □

Proof of Theorem 1.4. Let $\{p_1, \ldots, p_s\}$ be the set of irregular primes with respect to Euler polynomials. We will find an irregular prime not in this set.

For an integer k, set $n = k(p_1 - 1) \cdots (p_s - 1) - 1$. Since by [5, p. 232, Proposition 15.1.1(c)], $\left|\frac{B_{2m}}{2m}\right| \to \infty$ as $m \to \infty$, we may choose an integer k so large such that $\left|\frac{B_{n+1}}{n+1}\right| > 1$. Choose a prime p with $\mathrm{ord}_p\left(\frac{B_{n+1}}{n+1}\right) > 0$. By Claussen-von Staudt for Bernoulli numbers ([5, p. 233, Theorem 3]), we have $p - 1 \nmid n + 1$. Thus $p \neq p_i$, for $i = 1, \ldots, s$. Also $p \neq 2$. Since by [8, p. 10, Corollary 3.21],

$$E_n(0) = 2(1 - 2^{n+1}) \frac{B_{n+1}}{n+1}, \tag{2.4}$$

we have $\mathrm{ord}_p(E_n(0)) > 0$.

Now we show that p is irregular. Let $n \equiv l \pmod{p-1}$, where $0 \leq l < p-1$. Since $2 \mid p-1$, we have $n \equiv l \pmod 2$, thus l is also odd. From Kummer's congruence for Bernoulli numbers:

$$\frac{B_{n+1}}{n+1} \equiv \frac{B_{l+1}}{l+1} \pmod p$$

(see [10, p. 61, Corollary 5.14]), and Eq.(2.4) above, we have

$$E_n(0) \equiv E_l(0) \pmod p.$$

Notice that as assumption $\mathrm{ord}_p(E_n(0)) > 0$, we have $\mathrm{ord}_p(E_l(0)) > 0$, which shows that p is irregular. □

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